國立台中一中九十學年度合作杯數學競試詳解

1.解:令 $S_a = x_1x_2x_3$ ， $S_b = y_1y_2y_3$ $x_i, y_i \in \{a,b\}, i = 1,2,3$

設 $x_i < y_i$ 表 $x_i = a, x_j = b$

所求$=P((x_1x_2x_3) < (y_1y_2y_3))$

$= P(x_1 < y_1) + P(x_1 = y_1)\cdot P(x_2 < y_2) + P(x_1 = y_1)P(x_2 = y_2)P(x_3 < y_3)$

$= \frac{2}{3}\times\frac{2}{3} + \frac{4}{9}\times\frac{4}{9} + \frac{4}{9}\times\frac{4}{9}\times\frac{4}{9} = \frac{532}{729}$

註: $P(x_i < y_i) = \frac{2}{3}\times\frac{2}{3} = \frac{4}{9}$ ，

$P(x_i = y_i) = P(x_i = y_i = a) + P(x_i = y_i = b) = \frac{2}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{2}{3} = \frac{4}{9}$

2.解: $\forall n \in N, P_n(x) = P_{n-1}(x-n)$

$P_{20}(x) = P_{19}(x-20) = P_{18}(x-20-19) = P_{17}(x-20-19-18)$

$= \cdots\cdots =$

$= P_0(x-20-19-18-\cdots\cdots-2-1)$

$= P_0(x-210)$

$= (x-210)^3 + 313(x-210)^2 - 77(x-210) - 8$

$\therefore P_{20}(x)$ 中 x 項係數為 $3\cdot 210^2 - 313\cdot 420 - 77 = 763$

3.解:記 $f^{(m)}(n) = f(f(\cdots\cdots(f(n)\cdots))$

$\because 1000 = 90 + 7\times 130$

$\therefore f(90) = f^{(2)}(97) = f^{(3)}(104) = \cdots = f^{(131)}(1000)$

又 $f(1000) = 997$

$f(997) = f^{(2)}(1004) = f(1001) = 998$

$f(998) = f^{(2)}(1005) = f(1002) = 999$

$f(999) = f^{(2)}(1006) = f(1003) = 1000$

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$\therefore f^{(m)}(1000)$ 以 4 為週期輪流取 997,998,999,1000 而 $131 = 4\times 32 + 3$

$\therefore f(90) = f^{(131)}(1000) = f^{(3)}(1000) = 999$

4. 解:因為 $f(x)$必有因式 $x,(x-1),(x-2), \ldots\ldots\ldots,(x-25)$

令 $f(x)=x(x-1)(x-2)\cdots\cdots(x-25)\cdot q(x)$代入題設條件

$x(x-1)(x-2) \cdots\cdots(x-25)(x-26)\cdot q(x-1)=(x-26)\cdot x(x-1) \cdots\cdots(x-25)q(x)$

$\therefore q(x-1)=q(x)$即 $q(x)$表常數

$\therefore f(x)=A\cdot x(x-1)(x-2)\cdots\cdots(x-25)$其中 A 表異於 0 的常數

5. 解:遞迴式兩邊同除 $\sqrt{a_{n-1}\cdot a_{n-2}}$ 得 $\sqrt{\dfrac{a_n}{a_{n-1}}}-1=2\sqrt{\dfrac{a_{n-1}}{a_{n-2}}}$

令 $b_n=\sqrt{\dfrac{a_n}{a_{n-1}}},(n=1,2,\cdots)$ 則上式變爲 $b_n-1=2b_{n-1}$ 而 $b_1=1$

$\Rightarrow (b_n-\alpha)=2(b_{n-1}-\alpha)\therefore \alpha=-1$

$\Rightarrow b_n+1=2(b_{n-1}+1),n=2,3,4,\cdots,n$ 代入求積得

$b_n+1=2^n$ 即 $b_n=2^n-1,b_n^2=(2^n-1)^2$

$\therefore \dfrac{a_n}{a_{n-1}}=b_n^2,(n=1,2,\cdots)$, $n=2,3,\cdots,n$ 再代入求積得 $a_n=\prod_{k=1}^{n}(2^k-1)^2,(n\in N)$

6.證明: 設整數解爲 α

$\therefore \alpha-a_1,\alpha-a_2,\cdots\cdots,\alpha-a_{2m-1},\alpha-a_{2m}$ 相異

$P^{m(m-1)}=(-1)(1)(-p)(p)(-p^2)(p^2)\cdots(-p^{m-1})(p^{m-1})$

故 $\alpha-a_1,\alpha-a_2,\cdots\cdots,\alpha-a_{2m-1},\alpha-a_{2m}$ 爲

$-1,1,-p,p,\cdots\cdots,-p^{m-1},p^{m-1}$ 之排列

$\therefore (\alpha-a_1)+(\alpha-a_2)+\cdots\cdots+(\alpha-a_{2m-1})+(\alpha-a_{2m})$

$=(-1)+1+(-p)+p+\cdots\cdots+(-p^{m-1})+p^{m-1}=0$

$\therefore a_1+a_2+\cdots\cdots+a_{2m-1}+a_{2m}=2m\alpha,\alpha\in Z$ 爲 $2m$ 的倍數

7.解：設三邊長爲 $a\geq b\geq c$，$s=\dfrac{1}{2}(a+b+c)$，則

$s\times 1=\sqrt{s(s-a)(s-b)(s-c)}\ \Rightarrow\ 4(a+b+c)=(b+c-a)(c+a-b)(a+b-c)$

令 $x=\dfrac{1}{2}(b+c-a)$，$y=\dfrac{1}{2}(c+a-b)$，$z=\dfrac{1}{2}(a+b-c)$，

則 $x\leq y\leq z$ 且 $x+y+z=xyz$，但 $x+y+z=xyz\leq 3z\Rightarrow xy\leq 3$

(1)若 $xy=3$，則 $x+y+z=xyz=3z\ \Rightarrow\ x+y=2z\ \Rightarrow x=y=z$，此與 $xy=3$ 矛盾。

(2)若 $xy=2$，則 $x=1$，$y=2\ \Rightarrow z=3$。

(3)若 $xy=1$，則 $x=1$，$y=1$，$x+y+z=xyz\ \Rightarrow 2+z=z$，矛盾。

由(1)(2)(3)知 $x=1$，$y=2$，$z=3\ \Rightarrow a=3$，$b=4$，$c=5$。