第八次有獎徵答　解答

1.＜證明 1＞(2-25 張唐佑,2-19 張紘民,1-25 黃喬敬,1-4 齊凡翔,2-12 吳欣融,2-12 鄧石傑)

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c} \quad \Rightarrow \quad (a+b+c)(ab+bc+ca)=abc$

$\Rightarrow \quad a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+2abc=0$

$\Rightarrow \quad (a+b)(b+c)(c+a)=0$

$\Rightarrow \quad a+b=0$ 或 $b+c=0$ 或 $c+a=0$

設 $a+b=0$,則 $a=-b$,而 $a^{2n+1}=-b^{2n+1}$

$\therefore \quad \frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{c^{2n+1}}=\frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}$

同理 $b+c=0$, $c+a=0$ 時,上述也成立,得證

＜證明 2＞(1-26 陳晁偉,1-26 林虞軒)

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c} \quad \Rightarrow \quad \frac{1}{a}+\frac{1}{b}=\frac{1}{a+b+c}-\frac{1}{c} \quad \Rightarrow \quad \frac{a+b}{ab}=\frac{-(a+b)}{c(a+b+c)}$

$\Rightarrow \quad a+b=0$ 或 $-ab=c(a+b+c)$

$\Rightarrow \quad a+b=0$ 或 $c^2+ab+bc+ca=0$

$\Rightarrow \quad a+b=0$ 或 $(b+c)(c+a)=0$

$\Rightarrow \quad a+b=0$ 或 $b+c=0$ 或 $c+a=0$

設 $a+b=0$,則 $a=-b$,而 $a^{2n+1}=-b^{2n+1}$

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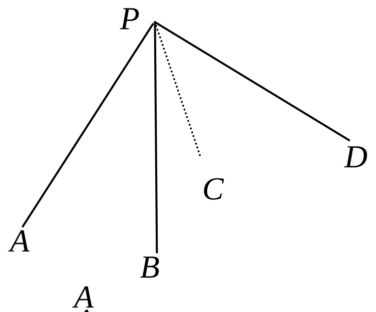
同理 $b+c=0$, $c+a=0$ 時,上述也成立,得證

2.＜解＞：(2-26 林志遠,2-25 蔡友倫 1-26 陳晁偉)

在四條線段上.分別取一些 A,B,C,D,使 $\overline{PA}=\overline{PB}=\overline{PC}=\overline{PD}$,因任兩線段之夾角均為 θ,

$\therefore$ $\triangle PAB \cong \triangle PBC \cong \triangle PCD \cong \triangle PAD \cong \triangle PAC \cong \triangle PBD$

$\therefore \quad \overline{AB}=\overline{BC}=\overline{CD}=\overline{AD}=\overline{AC}=\overline{BD}=a$

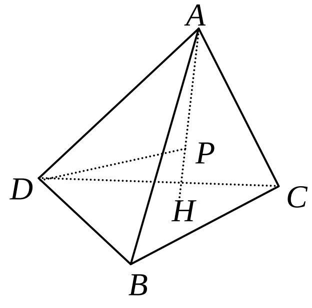


則 $A-BCD$ 為一正四面體,且 P 為四面體之外接球之球心

$\therefore \overline{AP}=\frac{3}{4}\overline{AH}$ （$\overline{AH}$ 為高）$=\frac{3}{4}\cdot\frac{\sqrt{6}}{3}a=\frac{\sqrt{6}}{4}a$

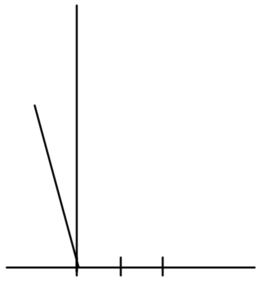
於 $\triangle PAD$ 中, $\angle APD=\theta, \overline{AD}=a, \overline{PA}=\overline{PD}=\frac{\sqrt{6}}{4}a$

$\therefore \cos\theta=\frac{(\frac{\sqrt{6}}{4})^2\cdot 2-1}{2\cdot\frac{\sqrt{6}}{4}\cdot\frac{\sqrt{6}}{4}}=-\frac{1}{3} \quad \Rightarrow \quad \tan\theta=-2\sqrt{2}$



3.＜解 1＞ (2-26 蕭雅澤)

作圖:1.作一角θ,使$\sin\theta=\frac{1}{3}$



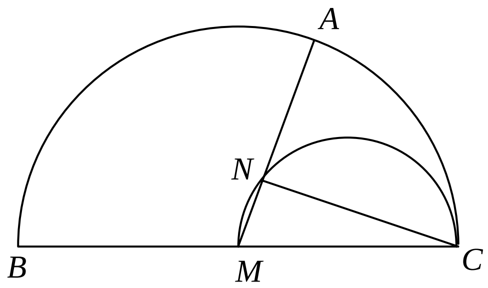
2.取$\overline{BC}=a$,作$\overline{BC}$中點M,分別以BC, MC為直徑作半圓BC, MC

3.作$\angle MCN=\theta$交半圓MC於N

4.連MN延長交半圓BC於A

5.連AB, BC,則$\triangle ABC$即為所求

證明: $\because BC$, MC為半圓$\therefore\triangle ABC$為Rt$\triangle$, $\angle MNC=90°$



$\therefore AM=BM=CM=\frac{a}{2}$,但$\sin\angle MCN=\frac{1}{3}$,

$\therefore MN=\frac{a}{6}\Rightarrow MN=\frac{1}{3}AM$

$\because AM$為中線,又$MN=\frac{1}{3}AM$

$\therefore N$為$\triangle ABC$重心 $\therefore CN$亦為中線

$\therefore\triangle ABC$合乎所求

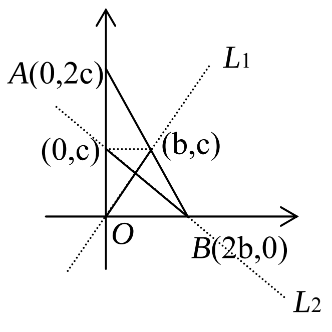
＜解 2＞ (1-26 陳晁偉)

1.如右圖建立座標系,找出b, c關係

由給予條件$L_1\perp L_2\quad\Rightarrow\quad m_1\cdot m_2=\frac{0-c}{0-b}\cdot\frac{c-0}{0-2b}=-1\quad\Rightarrow c^2=2b^2\Rightarrow b=\frac{c}{\sqrt{2}}$

$\therefore\overline{BO}=2b=\sqrt{2}\,c$, $\overline{AO}=2c$, $\overline{AB}=\sqrt{6}\,c$

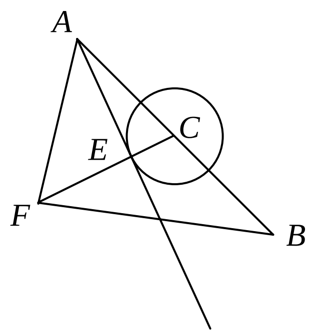
$\Rightarrow\overline{AB}:\overline{BO}:\overline{AO}=\sqrt{6}:\sqrt{2}:2=1:\sqrt{\frac{1}{3}}:\sqrt{\frac{2}{3}}$



2.由給予線段a,作出$\sqrt{\frac{1}{3}}a$, $\sqrt{\frac{2}{3}}a$

3.利用 a, $\sqrt{\frac{1}{3}}a$, $\sqrt{\frac{2}{3}}a$三線段即可作出所求Rt$\triangle$

＜解 3＞ (2-25 徐瑋璟)



[做法] 1.先作一線段$\overline{AB}$長為a,取其中點C

2.以C為圓心,$\frac{1}{6}a$為半徑作圓,再作$\overleftrightarrow{AD}$切此圓於E

3.過E作$\overline{CF}=\frac{1}{2}a$,連$AF$, BF, 則$\triangle AFB$即為所求

[證明]：$\because\overline{CA}=\overline{CB}=\overline{CF}$ $\therefore\triangle AFB$為Rt$\triangle$

直線$\overleftrightarrow{AD}$過重心E ($\because\overline{CE}:\overline{EF}=1:2$) $\therefore\overleftrightarrow{AD}$為中線且$\perp\overline{CF}$ $\therefore$有兩條中線互相垂直

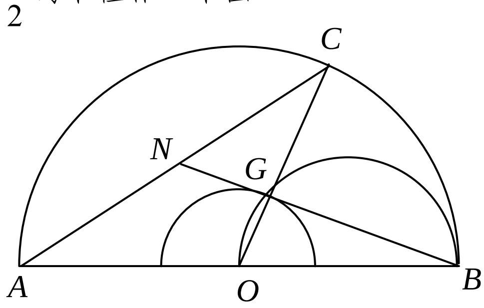
＜解 4＞ (1-26 林立人)

[做法]：1.作一線段 AB,長度為 a,取中點 O,以 O 為圓心 $\frac{a}{2}$ 為半徑作一半圓 Γ

2. 以 O 為圓心 $\frac{a}{6}$ 為半徑作一半圓 Γ_1

3. 以 $\overline{OB}$ 中點為圓心 $\frac{1}{2}\overline{OB}$ 為半徑作一半圓 Γ_2

4.設 Γ_1 與 Γ_2 交於一點 G,連接 OG 交 Γ 於點 C
則 $\triangle ABC$ 即為所求



[證明]：1.$\triangle ABC$ 中 $\angle ACB=90°$

2.$\overline{OC}$ 為一中線,且 $\overline{OG}=\frac{a}{6}=\frac{1}{3}\overline{OC}$ ∴G 為 $\triangle ABC$ 之重心

延長 BG 交 AC 於 N,則 $\overline{BN}$ 亦為一中線

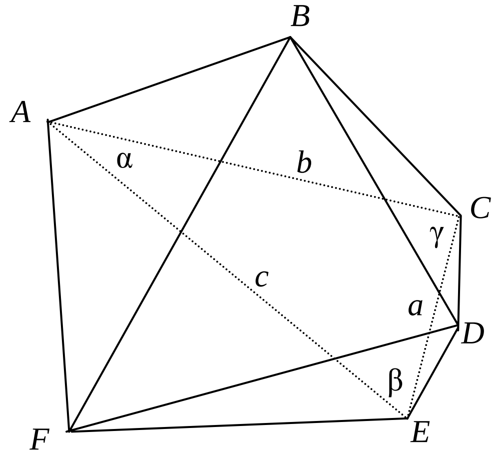
3.又 $\angle OGB=90°$ (半圓 Γ_2 之圓周角) ∴故兩中線 $\overline{BN}$,$\overline{OC}$ 互相垂直　故 $\triangle ABC$ 為所求

4.＜證 1＞$\triangle ACE$ 中,設 $\angle CAE=\alpha$,$\angle AEC=\beta$,$\angle ACE=\gamma$ 且 $\overline{CE}=a$,$\overline{AC}=b$,$\overline{AE}=c$

則 $\overline{AB}=\overline{BC}=\frac{b}{2}\sec 30°=\frac{b}{2}\times\frac{2\sqrt{3}}{3}=\frac{\sqrt{3}}{3}b$

($\because \angle ABC=120°$)

同理 $\overline{CD}.=\overline{DE}=\frac{\sqrt{3}}{3}a$,　$\overline{AF}=\overline{EF}=\frac{\sqrt{3}}{3}c$



$$\overline{BF}^2=\overline{AB}^2+\overline{AF}^2-2\overline{AB}\cdot\overline{AF}\cos(\alpha+60°)$$

$$=\frac{b^2}{3}+\frac{c^2}{3}-2\cdot\frac{\sqrt{3}}{3}b\cdot\frac{\sqrt{3}}{3}c\cdot(\frac{1}{2}\cos\alpha-\frac{\sqrt{3}}{2}\sin\alpha)$$

$$=\frac{1}{3}\left\{b^2+c^2-2bc(\frac{1}{2}\cdot\frac{b^2+c^2-a^2}{2bc}-\frac{\sqrt{3}}{2}\cdot\frac{a}{2R})\right\}$$,R 表 $\triangle ACE$ 外接圓半徑

$$=\frac{1}{3}\left\{b^2+c^2-\frac{b^2+c^2-a^2}{2}+\frac{\sqrt{3}abc}{R}\right\}=\frac{1}{3}\left\{\frac{a^2+b^2+c^2}{2}+\frac{\sqrt{3}abc}{R}\right\}$$

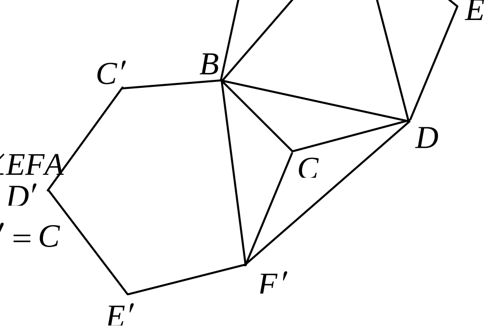
同理 $\overline{BD}^2=\overline{DF}^2=\frac{1}{3}\left\{\frac{a^2+b^2+c^2}{2}+\frac{\sqrt{3}abc}{R}\right\}$ ∴$\triangle BDF$ 為正三角形

＜證 2＞ (2-26 林志遠)

$\because \overline{AB}=\overline{BC}$,$\overline{CD}=\overline{DE}$,$\overline{EF}=\overline{FA}$,$\angle ABC=\angle CDE=\angle EFA$

∴將六邊形 $ABCDEF$ 旋轉得六邊形 $A'BC'D'E'F'$ 且 $A'=C$

於是△ $BF'D\cong\triangle BFD$, △ $BF'A'\cong\triangle BFA$



$\Rightarrow \angle ABF + \angle DBC = \angle A'BF' + \angle DBC = \angle F'BD + \angle FBD$

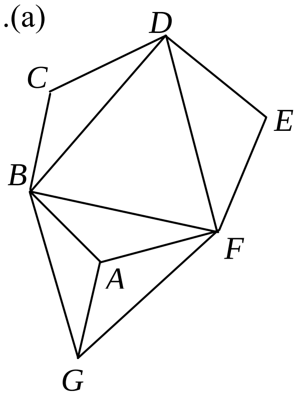
$\Rightarrow \angle ABC = 2\angle FBD = 120^\circ \quad \Rightarrow \angle FBD = 60^\circ$

同理$\angle BFD = \angle FBD = 60^\circ$ 故$\triangle BDF$為正三角形

＜證 3＞ (2-26 譚國棟)

(1)以 B 為頂點,$\overline{BA}$為一邊,作$\angle ABG = \angle DBC$ 且$\overline{BG} = \overline{BD}$,連$\overline{AG}$,又$\overline{AB} = \overline{BC}$

$\Rightarrow \triangle BAG \cong \triangle BCD$ (SAS) $\therefore \angle BAG = \angle C$, $\overline{AG} = \overline{CD}$ ….(a)



(2)連$\overline{FG}$,∵六邊形內角和720°

$\Rightarrow \angle C + \angle E + \angle BAF + 120^\circ + 120^\circ + 120^\circ = 720^\circ$

$\Rightarrow \angle C + \angle E + \angle BAF = 360^\circ$

又$\angle BAF + \angle BAG + \angle GAF = 360^\circ$

$\Rightarrow \angle GAF = \angle E$

又$\overline{AG} = \overline{CD} = \overline{DE}$, $\overline{EF} = \overline{AF}$

$\Rightarrow \triangle DEF \cong \triangle GAF \quad \therefore \overline{FG} = \overline{FD}$…….(b)

(3) $\because \overline{BF} = \overline{BF}$ 再由(a) (b) $\Rightarrow \triangle BFG \cong \triangle BFD \Rightarrow \angle FBD = \angle FBG = \angle ABF + \angle DBC$

又$\angle BFD + \angle ABF + \angle DBC = 120^\circ \Rightarrow \angle FBD = 60^\circ$

(4)同理$\angle BFD = \angle BDF = 60^\circ$

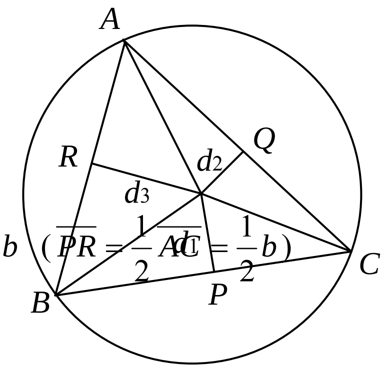
$\therefore \triangle BDF$為正三角形

5.＜證 1＞(1-26 林虞軒)

1.由圓心 O 作$\overline{BC}$, $\overline{CA}$, $\overline{AB}$的垂直線,垂足分別為 P,Q,R

則 P,Q,R 為各邊中點且$\overline{OP} = d_1$, $\overline{OQ} = d_2$, $\overline{OR} = d_3$

又設$\overline{BC} = a$, $\overline{CA} = b$, $\overline{AB} = c$, $S = \frac{1}{2}(a+b+c)$



2.因 O,P,B,R 四點共圓由托勒密定理知

$\overline{OP} \cdot \overline{BR} + \overline{OR} \cdot \overline{BP} = \overline{OB} \cdot \overline{PR} \quad \Rightarrow \frac{1}{2}c \cdot d_1 + \frac{1}{2}a \cdot d_3 = R \cdot \frac{1}{2}b \quad (\overline{PR} = \frac{1}{2}\overline{AC} = \frac{1}{2}b)$

$\Rightarrow \quad cd_1 + ad_3 = bR$…….(1)

同理 $\quad bd_1 + ad_2 = cR$…….(2)

$bd_3 + cd_2 = aR$…….(3)

3.$\triangle ABC = r\,S \Rightarrow \frac{1}{2}(ad_1 + bd_2 + cd_3) = r \cdot \frac{1}{2}(a+b+c) \Rightarrow ad_1 + bd_2 + cd_3 = (a+b+c) \cdot r$ …(4)

(1) + (2) + (3) + (4) $(a+b+c)(d_1+d_2+d_3) = (a+b+c)R + (a+b+c)r = (a+b+c)(R+r)$

$\therefore \ d_1 + d_2 + d_3 = R + r$

＜證 2＞(2-26 林志遠)

$\angle BOC = 2\angle A$, $\angle COA = 2\angle B$, $\angle AOB = 2\angle C$

$d_1 + d_2 + d_3 = R \cdot (\cos A + \cos B + \cos C)$

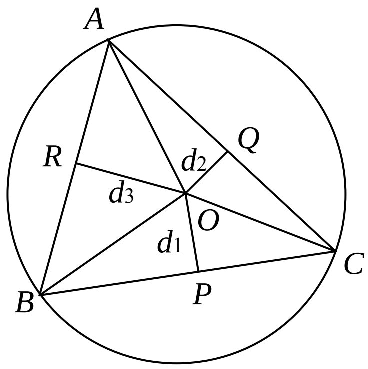
$$= R \cdot \frac{a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 - a^3 - b^3 - c^3}{2abc}$$

$$= R + R \cdot \frac{a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 - a^3 - b^3 - c^3 - 2abc}{2abc}$$

$$= R + \frac{abc}{2(a+b+c)r} \cdot \frac{a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 - a^3 - b^3 - c^3 - 2abc}{2abc}$$

$$= R + \frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{4(a+b+c)^2} \cdot \frac{1}{r} = R + \left(\frac{2\Delta}{a+b+c}\right)^2 \cdot \frac{1}{r} = R + r$$

[∵ $\triangle = \frac{abc}{4R}$ ， $S = \frac{1}{2}(a+b+c)$]



＜證 3＞(2-26 蕭雅澤)

∵O 為外接圓的圓心

∴$\angle BOC = 2\angle A$, $\angle COA = 2\angle B$, $\angle AOB = 2\angle C$

∴$d_1 = R\cos A$, $d_2 = R\cos B$, $d_3 = R\cos C$

$\Rightarrow$ $d_1 + d_2 + d_3 = R(\cos A + \cos B + \cos C)$

$$= R \cdot \left(2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2}\right)$$

$$= R \cdot \left(1 + 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 2\sin^2\frac{C}{2}\right) = R \cdot \left[1 + 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)\right]$$

$$= R \cdot \left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = R \cdot \left(1 + 4\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{(s-c)(s-a)}{ca}} \cdot \sqrt{\frac{(s-a)(s-b)}{ab}}\right)$$

$$= R \cdot \left(1 + 4\,\frac{(s-a)(s-b)(s-c)}{abc}\right) = R \cdot \left(1 + \frac{4}{abc} \cdot \frac{s(s-a)(s-b)(s-c)}{s}\right)$$

$$= R \cdot \left(1 + \frac{4}{abc} \cdot \frac{\Delta}{s} \cdot \Delta\right) = R \cdot \left(1 + 4r \cdot \frac{\frac{1}{2}bc\sin A}{abc}\right) = R \cdot \left(1 + 4r \cdot \frac{\frac{1}{2}\sin A}{2R\sin A}\right) = R \cdot \left(1 + \frac{r}{R}\right) = R + r$$