第 26 次金頭腦解答：

1. 【311 裘喻翔】

由根與係數：$\begin{cases} a+b+c=-a \cdots\cdots\cdots ① \\ ab+bc+ca=b \cdots\cdots ② \\ abc=-c \cdots\cdots\cdots\cdots\cdots ③ \end{cases}$

(1)若 $c=0$

由②：$ab=b$

a	b	c
0	0	0
1	$\neq 0$	0

若 $b\neq 0$ 時，$a=1$　代入①　$\Rightarrow$　$b=-2$

(2)若 $c\neq 0$　　由③：$ab=-1$　$\Rightarrow$　a、$b\neq 0$

若 $a=\pm 1$，則

a	b	(代入①)　c
1	-1	-1
-1	1	無解

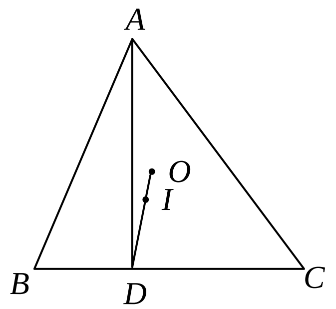
若 $a\neq\pm 1$，則 $a=-\dfrac{1}{b}$　代入②　得　$-1+c(a+b)=b$　$\Rightarrow$　$c=\dfrac{b+1}{a+b}$

由①　$2a+b+\dfrac{b+1}{a+b}=0$　($\because a$、$b\neq 0$　$\therefore a+b\neq 0$)

$\Rightarrow$　$2a^2-2-1+b^2+b+1=0$　$\Rightarrow$　$\dfrac{2}{b^2}+b^2+b-2=0$　$\Rightarrow$　$b^4+b^3-2b^2+2=0$

$\Rightarrow$　$(b+1)(b^3-b+2)=0$　$\therefore b$除-1 以外無有理解，但$a\neq\pm 1$，$\therefore$不合

2.



如圖，設 $\overline{AB}=c$, $\overline{BC}=a$, $\overline{CA}=b$ ，

且 $\overleftrightarrow{AI}$ 交$\triangle ABC$ 的外接圓於 K 點，

則 $\overline{OK}$ 是圓 O 的半徑記為 R，$\because \overline{OK}\perp\overline{BC}$ ，$\therefore \overline{OK}\,//\,\overline{AD}$

$\therefore \dfrac{\overline{AI}}{\overline{IK}}=\dfrac{\overline{AD}}{\overline{OK}}=\dfrac{c\cdot\sin B}{R}=2\sin B\cdot\sin C \cdots\cdots\cdots ①$

又

$\angle ABI=\angle IBC=\dfrac{1}{2}\angle B$, $\angle CBK=\angle CAK=\dfrac{1}{2}\angle A$, $\angle AKB=\angle ACB=\angle C$, $\angle BAK=\dfrac{1}{2}\angle A$

$$\therefore \frac{\overline{AI}}{\overline{IK}}=\frac{S_{\Delta ABI}}{S_{\Delta KBI}}=\frac{\frac{1}{2}c\cdot\overline{BI}\sin\frac{B}{2}}{\frac{1}{2}\overline{BK}\cdot\overline{BI}\sin\frac{A+B}{2}}=\frac{c}{\overline{BK}}\cdot\frac{\sin\frac{B}{2}}{\cos\frac{C}{2}}=\frac{\sin C}{\sin\frac{A}{2}}\cdot\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}=\frac{2\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}$$

由①、②得 $2\sin B\sin C=\dfrac{2\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}$

所以 $4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=1$

設△ABC的BC邊上的旁切圓半徑r_a，則 $\frac{1}{2}bc\cdot\sin A=S_{\Delta ABC}=\frac{1}{2}r_a(b+c-a)$

所以 $r_a=\dfrac{bc\cdot\sin A}{b+c-a}=2R\cdot\dfrac{\sin A\cdot\sin B\cdot\sin C}{\sin B+\sin C-\sin A}=2R\cdot\dfrac{\sin A\cdot\sin B\cdot\sin C}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}-2\sin\frac{B+C}{2}\cos\frac{B+C}{2}}$

$$=R\cdot\frac{\sin A\cdot\sin B\cdot\sin C}{\sin\frac{B+C}{2}\cdot2\sin\frac{B}{2}\cdot\sin\frac{C}{2}}=4R\cdot\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$=R$$

即△ABC的外接圓半徑R等於BC邊上旁切圓半徑r_a

3. 【123 蘇子銘】

若 $f(x)$、$g(x)$、$h(x)$爲二次方以上，則會有多解或重覆，不符題目。

∴設 $f(x)$ 、$g(x)$、$h(x)$爲一次式分別爲：$a(x+1)$，bx，$cx+d$

則 $a(x+1)+bx+cx+d=3x+2$

$a(x+1)-bx+cx+d=-2x+2$

$-a(x+1)+bx+cx+d=-1$

$$\Rightarrow\begin{cases}a+b+c=3\\a-b+c=-2\\-a+b+c=-1\end{cases}\quad and\quad\begin{cases}a+d=2\\-a+d=-1\end{cases}$$

$$a=\frac{3}{2}\quad b=\frac{5}{2}\quad c=-1\quad d=\frac{1}{2}\qquad\therefore\begin{cases}f(x)=\frac{3}{2}(x+1)\\g(x)=\frac{5}{2}x\\h(x)=-x+\frac{1}{2}\end{cases}$$

4. 【123 阮智楷】

$a_2 = 1 + a_1 \cdot \sin\theta$

$a_3 = 1 + a_2 \cdot \sin\theta = 1 + \sin\theta + \sin^2\theta = \dfrac{1-\sin^3\theta}{1-\sin\theta}$

$a_4 = 1 + \sin\theta + \sin^2\theta + \sin^3\theta = \dfrac{1-\sin^4\theta}{1-\sin\theta}$

$\vdots$

$a_n = 1 + \sin\theta + \sin^2\theta + \cdots + \sin^{n-1}\theta = \dfrac{1-\sin^n\theta}{1-\sin\theta}$

又$b_k = \cot\theta[1-(1-\sin\theta)a_k]^2$

$= \cot\theta[1-(1-\sin\theta)\cdot\dfrac{1-\sin^k\theta}{1-\sin\theta}]^2$

$= \cot\theta(1-1+\sin^k\theta)^2$

$= \cot\theta\cdot\sin^{2k}\theta$

$S_n = \sum_{k=1}^{n} b_k = \cot\theta\cdot\sin^2\theta + \cot\theta\cdot\sin^4 + \cdots + \cot\theta\cdot\sin^{2n}\theta$

$= \cot\theta(\sin^2\theta + \sin^4 + \cdots + \sin^{2n}\theta) = \cot\theta\cdot\dfrac{\sin^2\theta[1-(\sin^2\theta)^n]}{1-\sin^2\theta}$

$= \dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{\sin^2\theta[1-(\sin^2\theta)^n]}{\cos^2\theta} = \dfrac{\sin\theta - \sin^{2n+1}\theta}{\cos\theta} = \tan\theta - \tan\theta\cdot\sin^{2n}\theta$

$= \tan\theta(1-\sin^{2n}\theta)$

$\because -1 \le \sin\theta \le 1$　　故　$0 \le sin^2\theta \le 1$

①若 $0 \le \sin^2\theta < 1$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(\tan\theta - \tan\theta\cdot\sin^{2n}\theta)$

$= \tan\theta$

②若$\sin^2\theta = 1$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(\tan\theta - \tan\theta\cdot\sin^{2n}\theta) = 0$

5. $\begin{cases} x^2 + y^2 + \dfrac{2xy}{x+y} = 1 \cdots\cdots ① \\ \sqrt{x+y} = x^2 - y \cdots\cdots\cdots ② \end{cases}$

若 x、y 爲一組解，則 $x + y > 0$，

假設 $x + y > 1$，①$\Rightarrow 1 = x^2 + y^2 + \dfrac{2xy}{x+y} > \dfrac{x^2+y^2}{x+y} + \dfrac{2xy}{x+y} = \dfrac{(x+y)^2}{x+y} = x + y > 1$　$\rightarrow\leftarrow$

同理，假設 $x + y < 1$，①$\Rightarrow 1 = x^2 + y^2 + \dfrac{2xy}{x+y} < \dfrac{x^2+y^2}{x+y} + \dfrac{2xy}{x+y} = \dfrac{(x+y)^2}{x+y} = x + y < 1$　$\rightarrow\leftarrow$

$\therefore x + y = 1$ 代入②　得 $1 = x^2 - (1-x)$　$\Rightarrow x$的兩根爲 1 與 -2

$\therefore (x, y) = (1,0)$或$(-2,3)$