合作盃數學金頭腦第廿次有獎徵答詳解

1.令 $h(x)=g(x)-f(x)=x^3+(a-p)x^2+(b-q)x+c(c-r)$

$\because h(1)=g(1)-f(1)<0$

$h(2)=g(2)-f(2)>0$

$h(3)=g(3)-f(3)<0$

$h(\infty)>0$

由勘根定理知 $h(x)$=0,在(1,2),(2,3),(3, ∞)各有一正實根, $\alpha,\beta,\gamma>0$

$\Rightarrow \alpha\beta+\beta\gamma+\gamma\alpha=b-q>0$

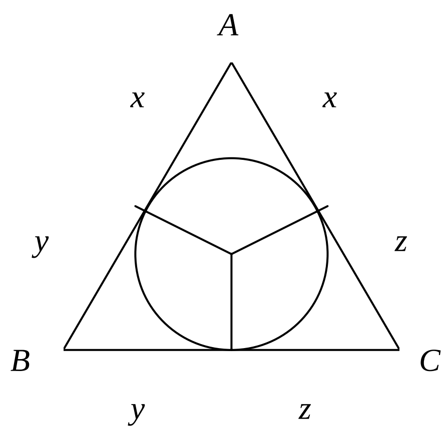
2.設 $\exists a,b,c,d,e,f\in C$ ，使得 $x^2+y^2+z^2=(ax+by+cz)\times(dx+ey+fz)$

比較係數可得 $abcdef\neq 0$

$$\begin{cases}ad=1\\ be=1\\ cf=1\end{cases}\Rightarrow\begin{cases}d=1/a\ldots(1)\\ e=1/b\ldots(2)\\ f=1/c\ldots(3)\end{cases}\qquad 且\begin{cases}bd+ae=0\ldots(4)\\ cd+af=0\ldots(5)\\ ce+bf=0\ldots(6)\end{cases}$$

(1),(2)代入(4)可得 $\frac{a^2+b^2}{ab}=0\Rightarrow a^2+b^2=0\Rightarrow a=b=0$ ∴矛盾

$$3.\begin{cases}a=y+z\\ b=x+z\\ c=x+y\end{cases}\Rightarrow a+b+c=2(x+y+z)$$



$$\Rightarrow s=\frac{a+b+c}{2}=x+y+z$$

$$rs=\sqrt{s(s-a)(s-b)(s-c)(s-d)}=\sqrt{(x+y+z)xyz}$$

$$\Rightarrow r=\frac{\sqrt{(x+y+z)xyz}}{x+y+z}=\sqrt{\frac{xyz}{x+y+z}}$$

欲證明： $\frac{1}{(s-a)^2}+\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}\geq\frac{1}{r^2}$

即證明： $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\geq\frac{x+y+z}{xyz}$

由柯西不等式： $(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2})(\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{x^2})\geq(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx})^2$

$$(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2})\geq(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx})=\frac{x+y+z}{xyz}$$

4.設 $S=a+b+c+d$

$$\text{原式}=\frac{S-(a+b+c)}{a+b+c}+4\times\frac{S-(a+b+d)}{a+b+d}+9\times\frac{S-(a+c+d)}{a+c+d}+16\times\frac{S-(b+c+d)}{b+c+d}$$

$$=S\left[\frac{1}{a+b+c}+\frac{4}{a+b+d}+\frac{9}{a+c+d}+\frac{16}{b+c+d}\right]-(1+4+9+16)$$

$$=\frac{1}{3}\left[(a+b+c)+(a+b+d)+(a+c+d)+(b+c+d)\right]$$

$$\times\left[\frac{1}{a+b+c}+\frac{4}{a+b+d}+\frac{9}{a+c+d}+\frac{16}{b+c+d}\right]-30$$

$$=\frac{1}{3}\left[\left(\sqrt{a+b+c}\right)^2+\left(\sqrt{a+b+d}\right)^2+\left(\sqrt{a+c+d}\right)^2+\left(\sqrt{b+c+d}\right)^2\right]$$

$$\times\left[\left(\frac{1}{\sqrt{a+b+c}}\right)^2+\left(\frac{2}{\sqrt{a+b+d}}\right)^2+\left(\frac{3}{\sqrt{a+c+d}}\right)^2+\left(\frac{4}{\sqrt{b+c+d}}\right)^2\right]-30$$

由柯西不等式

$$\geq\frac{1}{3}\left(\sqrt{a+b+c}\times\frac{1}{\sqrt{a+b+c}}+\sqrt{a+b+d}\times\frac{1}{\sqrt{a+b+d}}+\sqrt{a+c+d}\times\frac{1}{\sqrt{a+c+d}}+\sqrt{b+c+d}\times\frac{1}{\sqrt{b+c+d}}\right)^2-30$$

$$=\frac{1}{3}(1+2+3+4)^2-30=\frac{10}{3}$$

5. ΔABC 中，令 $\tan A=\alpha,\tan B=\beta,\tan C=\gamma$

$\Rightarrow \tan A\tan B\tan C=\tan A+\tan B+\tan C$

$\Rightarrow \alpha\beta\gamma=\alpha+\beta+\gamma=-\frac{1}{6}$

$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\alpha\gamma)$

$=(\alpha+\beta+\gamma)[(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\alpha\gamma)]$

$\Rightarrow -\frac{181}{216}-3\times(-\frac{1}{6})=(-\frac{1}{6})[(-\frac{1}{6})^2-3(\alpha\beta+\beta\gamma+\alpha\gamma)]$

$\Rightarrow \alpha\beta+\beta\gamma+\alpha\gamma=-\frac{2}{3}$

$\Rightarrow \alpha$， β， γ 爲 $6x^3+x^2-4x+1=0$ 之三根

$\because 6x^3+x^2-4x+1=(x+1)(2x-1)(3x-1)=0$

$\Rightarrow \tan A,\tan B,\tan C$ 爲 $-1,\frac{1}{2},\frac{1}{3}$

A,B,C 中最大角爲 $\tan\theta=-1$

$\Rightarrow \theta=135°$