合作盃數學金頭腦第十九次有獎徵答詳解

1、 $a_{n+1}=a_n+\frac{1}{a_n}\Rightarrow a_{n+1}^2=a_n^2+\frac{1}{a_n^2}+2\geq 2\sqrt{a_n^2\cdot\frac{1}{a_n^2}}+2=4\Rightarrow\frac{1}{a_{n+1}^2}\leq\frac{1}{4}$

$$\sum_{n=1}^{99}a_{n+1}^2=\sum_{n=1}^{99}(a_n^2+\frac{1}{a_n^2}+2)$$

$$\Rightarrow a_{100}^2=a_1^2+\frac{1}{a_1^2}+\sum_{n=2}^{99}\frac{1}{a_n^2}+2\times 99=1+1+\sum_{n=2}^{99}\frac{1}{a_n^2}+198=200+\sum_{n=2}^{99}\frac{1}{a_n^2}$$

$$\therefore 200<a_{100}^2<200+98\times\frac{1}{4}=224.5\ \therefore [a_{100}]=14$$

2、(1)由柯西不等式：

$$(\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c})\cdot$$
$$[a(b+2c+3d)+b(c+2d+3a)+c(d+2a+3b)+d(a+2b+3c)]\geq(a+b+c+d)^2$$

$$\Rightarrow(\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c})\cdot$$
$$[4(ab+ac+ad+bc+bd+cd)]\geq(a+b+c+d)^2$$

$$(\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c})\geq\frac{(a+b+c+d)^2}{4(ab+ac+ad+bc+bd+cd)}$$

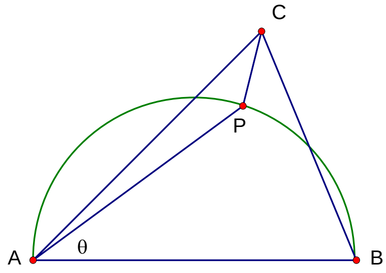
(2)由 $(a-b)^2+(a-c)^2+(a-d)^2+(b-c)^2+(b-d)^2+(c-d)^2\geq 0$

$$\Rightarrow a^2+b^2+c^2+d^2\geq\frac{2}{3}(ab+ac+ad+bc+bd+cd)$$

$$\Rightarrow(a+b+c+d)^2\geq\frac{8}{3}(ab+ac+ad+bc+bd+cd)$$

(3)由(1)(2)得所求 $\geq\frac{(a+b+c+d)^2}{4(ab+ac+ad+bc+bd+cd)}\geq\frac{1}{4}\times\frac{8}{3}=\frac{2}{3}$

3、$\overline{AP}=\overline{AB}\cdot\cos\theta=\cos\theta$，$0\le\theta\le\frac{\pi}{2}$



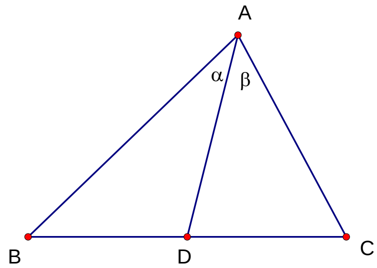
$\therefore\triangle$APC=$\frac{1}{2}\overline{AC}\cdot\overline{AP}\cdot|\sin(\frac{\pi}{4}-\theta)|$

$=\frac{1}{2}\cdot 1\cdot\cos\theta\cdot|\sin\frac{\pi}{4}\cos\theta-\cos\frac{\pi}{4}\sin\theta|$

$=\frac{\sqrt{2}}{8}\cdot|1+\sqrt{2}\sin(\frac{\pi}{4}-2\theta)|\le\frac{\sqrt{2}}{8}(1+\sqrt{2}\cdot\frac{\sqrt{2}}{2})=\frac{\sqrt{2}}{4}$

$\therefore$當 θ =0$\Rightarrow$MAX=$\frac{\sqrt{2}}{4}$

4、由已知$\frac{\overline{BD}}{\overline{AD}}\cdot\frac{\overline{CD}}{\overline{AD}}=\frac{1}{2}$



$\because$正弦定理$\Rightarrow\frac{\overline{BD}}{\overline{AD}}\cdot\frac{\overline{CD}}{\overline{AD}}=\frac{\sin\alpha}{\sin B}\cdot\frac{\sin\beta}{\sin C}=\frac{1}{2}$

$\therefore\sin B\sin C+\cos A$

$=2\sin\alpha\sin\beta+\cos(\alpha+\beta)$

$=2\sin\alpha\sin\beta+\cos\alpha\cos\beta-\sin\alpha\sin\beta$

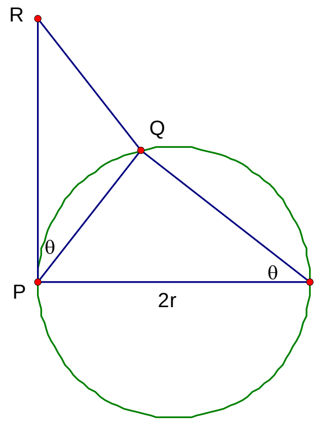
$=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$=\cos(\alpha-\beta)$

≤ 1

得證.

5、$\triangle$PQR=$\frac{1}{2}\overline{PQ}\cdot\overline{QR}\sin(\pi-2\theta)$



$=\frac{1}{2}\cdot 4r^2\sin^2\theta\cdot\sin 2\theta=r^2(1-\cos 2\theta)\sin 2\theta$

設 $f(\theta)=(1-\cos 2\theta)\sin 2\theta$

解 $f'(\theta)=0$ 得 $\cos 2\theta=-\frac{1}{2}$，$\theta=\frac{\pi}{3}$

有面積最大$\frac{3\sqrt{3}}{4}r^2$