第十八 金頭腦解答

1

$\because a_1, a_2, a_3, \ldots\ldots, a_n$ 爲 $f(x) = x^n + x + 1 = 0$ 之根

$\Rightarrow a_1^n + a_1 + 1 = 0 \ldots\ldots\ldots\ (1)$

$a_2^n + a_2 + 1 = 0 \ldots\ldots\ldots(2)$

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$a_n^n + a_n + 1 = 0 \ldots\ldots\ldots(n)$

$(1) + (2) + \ldots\ldots + (n) \Rightarrow a_1^n + a_2^n + \ldots\ldots + a_n^n + (a_1 + a_2 + \ldots\ldots + a_n) + n = 0$

由根與係數知 $\Rightarrow a_1^n + a_2^n + \ldots\ldots + a_n^n = -n \ldots\ldots\ldots\ \ldots\ldots\ <1>$

$\because x^n = -x - 1 \Rightarrow x^{n-1} = -1 - \dfrac{1}{x} \Rightarrow a_i^{n-1} = -1 - \dfrac{1}{a_i}$

$\Rightarrow a_1^{n-1} + a_2^{n-1} + \ldots\ldots + a_n^{n-1} = -n - (\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots\ldots + \dfrac{1}{a_n})$

而 $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \ldots\ldots, \dfrac{1}{a_n}$ 爲 $f(\dfrac{1}{x}) = x^n + x^{n-1} + 1 = 0$ 之一根

$\Rightarrow a_1^{n-1} + a_2^{n-1} + \ldots\ldots + a_n^{n-1} = -n - (-1) = -n + 1 \ldots\ldots\ldots\ \ldots\ldots\ <2>$

$<2> - <1> \Rightarrow$ 得証

2

(1) $\cos\theta$ 是有理數

(2) 證明

$(\cos 3\theta + i\sin 3\theta)(\cos 5\theta + i\sin 5\theta) = \cos 8\theta + i\sin 8\theta$

$\Rightarrow (\cos 3\theta \cos 5\theta - \sin 3\theta \sin 5\theta) = \cos 8\theta \Rightarrow \sin 3\theta \sin 5\theta$

$= \cos 3\theta \cos 5\theta - \cos 8\theta$

爲有理數

$\because \cos\theta = \cos(6\theta - 5\theta)$

$= \cos 6\theta \cos 5\theta + \sin 6\theta \sin 5\theta$

$= (2\cos^2 3\theta - 1)\cos 5\theta + 2\cos 3\theta \sin 3\theta \sin 5\theta$

爲有理數

3

由算幾不等式知

$$(b^3-1)+(b^3-1)+(b^3-1)+\frac{3}{2}+\frac{3}{2} \ \geq \ 5\sqrt[5]{(b^3-1)^3\times\frac{9}{4}}$$

$$\therefore \quad \frac{3b^3}{5} \ \geq \ \sqrt[5]{(b^3-1)^3\times\frac{9}{4}}$$

$$\Rightarrow \ (\frac{3}{5})^5\times b^{15} \ \geq \ (b^3-1)^3\times\frac{9}{4} \ \Rightarrow \ (b^3-1)^3 \ \leq \ \frac{4}{9}(\frac{3}{5})^5 b^{15}$$

$$\Rightarrow \ b^3-1 \ \leq \ \frac{3}{5}\times\sqrt[3]{\frac{4}{25}}\times b^5 \ \Rightarrow \ \frac{1}{b^3-1} \ \geq \ \frac{5}{3}\times\sqrt[3]{\frac{25}{4}}\times\frac{1}{b^5}$$

$$\Rightarrow \ \frac{a^5}{b^3-1} \ \geq \ \frac{5}{3}\times\frac{\sqrt[3]{50}}{2}\times\frac{a^5}{b^5}\ldots\ldots\ldots\ldots(1)$$

同理 $\frac{b^5}{c^3-1} \ \geq \ \frac{5}{3}\times\frac{\sqrt[3]{50}}{2}\times\frac{b^5}{c^5}\ldots\ldots\ldots\ldots(2)$

$$\frac{c^5}{a^3-1} \ \geq \ \frac{5}{3}\times\frac{\sqrt[3]{50}}{2}\times\frac{c^5}{a^5}\ldots\ldots\ldots\ldots(3)$$

$(1)+(2)+(3)$

$$\Rightarrow \frac{a^5}{b^3-1}+\frac{b^5}{c^3-1}+\frac{c^5}{a^3-1} \ \geq \ \frac{5\sqrt[3]{50}}{6}(\frac{a^5}{b^5}+\frac{b^5}{c^5}+\frac{c^5}{a^5}) \ \geq \ \frac{5\sqrt[3]{50}}{6}\times 3\times(\frac{a^5}{b^5}+\frac{b^5}{c^5}+\frac{c^5}{a^5})$$

$$\Rightarrow \frac{a^5}{b^3-1}+\frac{b^5}{c^3-1}+\frac{c^5}{a^3-1} \ \geq \ \frac{5\sqrt[3]{50}}{2}$$

等號成立於 $a=b=c=\sqrt[3]{\frac{5}{2}}$

4

$|Z_1|=|Z_1+Z_2|=3$， $|Z_1+Z_2|=3\sqrt{3}$

$$\Rightarrow \ (Z_1+Z_2)(\overline{Z_1+Z_2})=9 \quad \Rightarrow \quad |Z_1|^2+Z_1\overline{Z_2}+\overline{Z_1}Z_2+|Z_2|^2=9\ldots\ldots..(1)$$

$$(Z_1-Z_2)(\overline{Z_1-Z_2})=27 \quad\quad |Z_1|^2-Z_1\overline{Z_2}-\overline{Z_1}Z_2+|Z_2|^2=27\ldots\ldots.(2)$$

$(1)-(2) \Rightarrow Z_1\overline{Z_2}-\overline{Z_1}Z_2=-9$，又 $|Z_1|=|Z_2|=3$

$\Rightarrow Z_1\overline{Z_2}+\overline{Z_1}Z_2=18\cos\theta=-9 \Rightarrow Z_1\overline{Z_2}=9W$ 或 $9W^2$

$$\Rightarrow \log_3\left|(Z_1\overline{Z_2})^{2000}+(\overline{Z_1}Z_2)^{2000}\right|=\log_3\left|3^{4000}(W^{2000}+W^{4000})\right|$$

$$\log_3 3^{4000}=4000$$

5

$n$邊形之對角線有$\frac{n(n-3)}{2}$條

$\Rightarrow \frac{n(n-3)}{2}=n \quad \Rightarrow \quad n=5$

故即求證五邊形之周長 p,對角線長和 $=q$

令邊長 $=$ a,則 p $=$ 5a,對角線長 $=$ b

$\frac{\mathrm{b}}{\sin 108^\circ}=\frac{a}{\sin 36^\circ}$

$\Rightarrow b=\frac{a\cos 18^\circ}{2\sin 18^\circ\cos 18^\circ}=\frac{a}{2\times\frac{\sqrt{5}+1}{4}}=\frac{\sqrt{5}+1}{2}a$

$\Rightarrow$ 對角線長和 $q=5b=\frac{5(\sqrt{5}+1)}{2}$

$\frac{q}{p}=\frac{\frac{5\left(\sqrt{5}+1\right)a}{2}}{5a}=\frac{\sqrt{5}+1}{2},\frac{p}{q}=\frac{2}{\sqrt{5}+1}=\frac{\sqrt{5}-1}{2}$

$\Rightarrow \frac{q}{p}-\frac{p}{q}=1$